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Why Does the Geometric Product Simplify the Equations of Physics?

Yaakov Friedman[1,2] and Yuriy Gofman[1]

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In the last decades it was observed that Clifford algebras and geometric product provide a model for different physical phenomena. We propose an explanation of this observation based on the theory of bounded symmetric domains and the algebraic structure associated with them. The invariance of physical laws is a result of symmetry of the physical world that is often expressed by the symmetry of the state space for the system implying that this state space is a symmetric domain. For example, the ball of all possible velocities is a bounded symmetric domain. The symmetry on this ball follow from the symmetry of the space-time transformations between two inertial systems, which fixes the so-called "symmetric velocity" between them. The Lorenz transformations acts on the ball S of symmetric velocities by conformal transformations. The ball S is a spin ball (type IV in Cartan's classification). The Lie algebra of this ball is defined a triple product that is closely related to geometric product. The relativistic dynamic equations in mechanics and for the Lorenz force is described by this Lie algebra and the triple product.

KEY WORDS: bounded symmetric domain; special relativity; geometric product; spin factor.

1. INTRODUCTION

In the last decades geometric product became an efficient tool for description of different areas in physics (see Baylis, 1996; Lasenby *et al.*, 2000). But why does the Clifford algebra and geometric product turned out to be so efficient in describing physical phenomena? We propose here an explanation of this observation based on the theory of bounded symmetric domains and the algebraic structure associated with them.

This explanation is based on our believe that unbounded and bounded symmetric domains and JB*-triple product associated with these domains may provide a model for different areas in physics. A bounded domain D in a Banach space

[1] Jerusalem College of Technology, Jerusalem, Israel.
[2] To whom correspondence should be addressed at Jerusalem College of Technology, P.O. Box 16031, 91160 Jerusalem, Israel; e-mail: friedman@mail.jct.ac.il.

is called a *bounded symmetric domain* if for every $z \in D$ there exists a smooth automorphism $s_z \in Aut(D)$ of period two on D, having z as the only fixed point. The smoothness of the automorphism may mean complex analytic, conformal (preserving angles) or projective (preserving linear segments) transformations. It is known that a domain D is a bounded symmetric domain if it has a symmetry about one point and is homogeneous in sense that for any two points $z, w \in D$ there is an automorphism $\varphi \in Aut(D)$ such that $\varphi(z) = w$.

There is a triple product uniquely associated with any bounded symmetric domain D in a Banach space A over C, obtained as follows (see Loos, 1977 for details): By fixing any point in D (we may assume for simplicity that this is the zero point of A) we may decompose $Aut(D)$ into rotations and translations. This implies that the Lie algebra $aut(D)$ will be a direct sum of linear terms, as generators of rotations, and generators of translations. It was shown (see Kaup, 1983; Loos, 1977) that the generators of translations are of the form

$$\xi_a(z) = a - q_a(z), \quad z \in A \tag{1}$$

where $q_a(z)$ is quadratic in z and is conjugate linear in a.

The quadratic form $q_a(z)$ could be rewritten as a real trilinear form $q_a(z) = \{z, a, z\}$ and by linearization of the quadratic dependence in z (polarization) we define a triple product on A with the following properties:

(i) $\{a, b, c\}$ is linear in a, c and conjugate linear in b,
(ii) $\{a, b, c\} = c, b, a$.

The operator $D(a, b)$ defined by

$$D(a, b)c = \{a, b, c\}, \quad c \in A \tag{2}$$

satisfies:

(iii) the operator $D(a, a)$ is a hermitian with positive spectrum,
(iv) for any $a, x, y, z \in A$

$$D(a, a)\{x, y, z\} = \{D(a, a)x, y, z\} - \{x, D(a, a)y, z\} + \{x, y, D(a, a)z\} \tag{3}$$

and
(v) $\|a\|^3 = \|\{a, a, a\}\|$.

A Banach space A with a triple product satisfying the above properties is called a *JB*-triple*. In Kaup (1983) it is shown that the category of bounded symmetric domains with respect to analytic maps is equivalent to the category of JB*-triples.

Any bounded symmetric domain could be decomposed into indecomposable domains, called Cartan factors. There are six types of Cartan factors and the JB*-triples associated with them (Dang and Friedman, 1987; Loos, 1977). The type 1 consists of bounded operators between two Hilbert spaces. The types 2 and 3 consist of antisymmetric and symmetric complex matrices, respectively, representing bounded operators on a Hilbert space. In all above types the triple product is defined by

$$\{a, b, c\} = \frac{ab^*c + cb^*a}{2}, \tag{4}$$

where b^* is the adjoint operator to b. The types 5 and 6 are exceptional of dimensions 16 and 27 respectively, and we will not be concerned with them here.

The Cartan factor of type 4, called also the *spin factor*, will be mainly used here. This factor is defined as the space R^n or C^n with the triple product

$$\{a, b, c\} = \langle a \mid b \rangle c + \langle c \mid b \rangle a - \langle a \mid \overline{c} \rangle \overline{b}. \tag{5}$$

As it was shown in Friedman and Russo (1986) any JB*-triple is isomorphic to the sum of an exceptional part and a subspace of operators on a Hilbert space with the triple product (4).

Why are the bounded symmetric domains and JB*-triples a good model for physics? Any law in physics must satisfy the symmetry or invariance principle. This principle states that the law should not change if we change the point from which we observe the phenomena. Such a principle imposes homogeneity of the state space of the system, that could be expressed as a symmetry of the domain representing the possible states of the system. This suggests that the state space is a symmetric domain. In order to be able to obtain numerical results we need an algebraic structure. Currently there is only one algebraic structure that has an origin only in geometry—the JB*-triple product.

For quantum systems the state space must poses some geometry coming from the measuring process. In Friedman and Russo (1992) and Friedman and Russo (1993) it was shown that this geometry implies that the state space of the system is a bounded symmetric domain. As it was noticed Günaydin (1980), all physically meaningful quantities in Quantum Mechanics depend only on the Jordan *triple* product rather than on the binary one. In Friedman and Russo (2001) it was shown that the canonical anticommutation relations could be efficiently represented on the spin factor and the Lorentz group is represented on this factor by spin 1 and spin $1/2$ representations.

In Friedman and Naimark (1992) and Friedman (1994) it was shown that bounded symmetric domains occur in transmission line theory and in special relativity. In the next section we will show how the principle of relativity lead to existence of a symmetry on the space-time continuum. From this symmetry alone we shall derive the Lorentz transformations and show that the possible velocities form a ball that is a bounded symmetric domain. The axis of the above symmetry is defined by so-called symmetric velocity.

In Section 4 we will show that the ball S of all possible symmetric velocities is a bounded symmetric domain with respect to the conformal group and is Cartan factor of type 4, called spin factor. The Lie algebra $aut(S)$ is described in terms of spin triple product. In Section 5 we will show that the dynamic equations in mechanic and for the electromagnetic forces with respect to the symmetric velocity are expressed through the spin triple product. In Section 6 we will show that there is a natural representation of the geometric product as operators on the spin factor and discuss the difference between this representation and its representation in the Clifford algebra.

2. RELATIVISTIC LINEAR SPACE-TIME TRANSFORMATIONS BASED ON SYMMETRY

In this section we will show how the Lorentz space-time transformations could be obtained from a symmetry without assumption of constancy of speed of light. This symmetry is a direct consequence of the Relativity Principle.

The basic tool of special relativity is the relation between the space-time descriptions of events in two inertial frames that is expressed by the Lorentz transformation. Einstein's original axiomatic derivation (Einstein, 1905) of Lorentz transformation formulation is based on two postulates: (i) the special relativity principle and (ii) the hypothesis of the constancy of speed of light in all inertial frames. A lot of work was done to show that the Lorentz transformations could be deduced from weaker assumptions starting from 1910 till now, see Schwartz (1984) and references therein.

The first Newton law states that in any inertial system an object moves with constant velocity if there are no forces acting on it. Such a motion is called a free motion and is described by straight lines in the space-time continuum. Thus, space-time transformations between two inertial system will preserve straight lines. We restrict ourself to inertial frames with the same space origin at time $t = 0$. By a known theorem in mathematics this implies that the space-time transformation between two inertial frames is a linear map. For another proof of linearity of the space-time transformations based on the assumption of homogeneity was given in Eisenberg (1967). By a simple argument this implies that the map between and is also a linear map that could be described by a matrix.

Consider now two inertial frames K and K′ with coordinates $\binom{t}{r}$ and $\binom{t'}{r'}$ respectively. We assume that the frames have the same origin and the two clocks at each origin synchronized at time $t = 0$. Moreover, we assume that the space axes are reversed as in Fig. 1. The reversion of the space axis is needed to preserve the symmetry, resulting from the principle of relativity, of the transformations between two inertial systems. Note that with this choice of the axes the velocity of O' in K is equal to the velocity of O in K′ and thus the transformation problem is fully symmetric with respect to K and K′. We will denote this

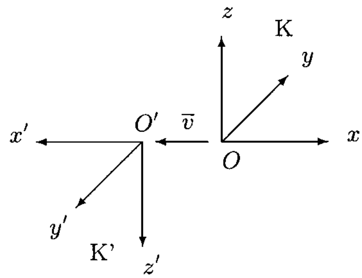

Fig. 1. Two symmetric reference frames.

transformation $S_{\bar{v}}$, since it is a symmetry and depends only on the velocity between the systems.

To describe the relative position between these systems we consider an event that occurs at O', corresponding to $\bar{r}' = 0$ at time t and express its position $\bar{r}$ in K. Since the system K′ moves with uniform velocity $\bar{v}$ with respect to K, this means that

$$\bar{r} = \bar{v}t. \tag{6}$$

Note that for this description we used as input position of an event in system K′ with time in K to calculate its position in K.

The space-time transformation between these frames could be considered as a "two-port linear black box" transformation with two inputs and two outputs. To be consistent with the description of relative position between the systems, we choose one of the inputs as a scalar t—the time of the event in K and the other input as a three-dimensional vector $\bar{r}'$ describing the position of the event in K′. Then one of the outputs is a scalar t'—the time of the event in K′ and the other output is a three-dimensional vector $\bar{r}$ describing the position of the event in K. The four components of the transformation $S_{\bar{v}}$ defined from

$$\begin{pmatrix} t' \\ \bar{r} \end{pmatrix} = S_{\bar{v}} \begin{pmatrix} t \\ \bar{r}' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} t \\ \bar{r}' \end{pmatrix} \tag{7}$$

will be denoted by S_{ij} for $i, j \in \{0, 1\}$ as in Fig. 2.

We describe now the meaning of the four linear maps occurring in the black box. To define the maps S_{21} and S_{11} consider an event that occurs at O', corresponding to $\bar{r}' = 0$, at time t in K, then $S_{21}(t)$ express the position of this

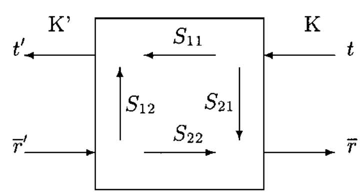

Fig. 2. Black box model for space-time transformations.

event in K and $S_{11}(t)$ express the time of this event in K′. Obviously, S_{21} describes, the relative velocity of frame K′ with respect to K and it is given by

$$S_{21}(t) = \overline{v}t \tag{8}$$

and S_{11} describes the time measured by the clock positioned at O' of an event occurring at O' at the time t in K and is given by

$$S_{11}(t) = \alpha t \tag{9}$$

for some constant α.

To define the maps S_{12} and S_{22} we will consider an event occurring at time $t = 0$ in K in space position $\overline{r'}$ in K′. Then $S_{12}(\overline{r'})$ will be the time of this event in K′ and $S_{22}(\overline{r'})$ will be the position of this event in K. Note that $S_{12}(\overline{r'})$ is also the time difference of two clocks both positioned at time $t = 0$ at a space point described by $\overline{r'}$ in K′, where the first one was synchronized to the clock at the common origin of the two systems within the frame K′ and the second one was synchronized to the clock at the origin within the frame K. Thus S_{12} describes the nonsimultaneity in K′ of simultaneous events in K with respect to their space displacement in K′ following from the difference in synchronization of clocks in K and K′. Since S_{12} is a linear map from R^3 to R, it is given by:

$$S_{12}(\overline{r'}) = \langle \overline{e} \mid \overline{r'} \rangle = \overline{e}^{\mathrm{T}} \cdot \overline{r'}, \tag{10}$$

for some vector $\overline{e} \in R^3$ where $\overline{e}^{\mathrm{T}}$ denotes the transpose of $\overline{e}$, the bracket $\langle | \rangle$ denotes the dot product in R^3 and(·)denotes matrix multiplication. Note that since the space is isotropic and the configuration of our systems has one unique divergent direction $\overline{v}$, therefor $\overline{e}$ is collinear to $\overline{v}$. Thus

$$\overline{e} = e\overline{v} \tag{11}$$

for some constant e.

Finally, the map S_{22} describes the transformation of the space displacement in K of simultaneous events in K with respect to their space displacement in K′ and it is given by:

$$S_{22}(\overline{r'}) = A\overline{r'} \tag{12}$$

for some 3×3 matrix A.

Our black box transformation can now be described by a 4×4 matrix $S_{\overline{v}}$ with block matrix entries from (8) to (10) and (12) as

$$\begin{pmatrix} t' \\ \overline{r} \end{pmatrix} = S_{\overline{v}} \begin{pmatrix} t \\ \overline{r'} \end{pmatrix} = \begin{pmatrix} \alpha & \overline{e}^{\mathrm{T}} \\ \overline{v} & A \end{pmatrix} \begin{pmatrix} t \\ \overline{r'} \end{pmatrix}. \tag{13}$$

If we now interchange the roles of systems K and K′, we will get a matrix $S'_{\overline{v}}$:

$$\begin{pmatrix} t \\ \overline{r'} \end{pmatrix} = S'_{\overline{v}} \begin{pmatrix} t' \\ \overline{r} \end{pmatrix} = \begin{pmatrix} \alpha' & \overline{e'}^{\mathrm{T}} \\ \overline{v'} & A' \end{pmatrix} \begin{pmatrix} t' \\ \overline{r'} \end{pmatrix}. \tag{14}$$

But the principle of relativity imply that switching the roles of K and K′ is non-recognizable. Hence

$$\alpha = \alpha', \qquad \overline{e}^{\mathrm{T}} = \overline{e'}^{\mathrm{T}}, \qquad \overline{v} = \overline{v'}, \qquad A = A'.$$

By combining (13) and (14) we get $S_{\overline{v}}^2 = I$, the identity, implying that $S_{\overline{v}}$ is a symmetry operator, that is,

$$\begin{pmatrix} \alpha & \overline{e}^{\mathrm{T}} \\ \overline{v} & A \end{pmatrix} \begin{pmatrix} \alpha & \overline{e}^{\mathrm{T}} \\ \overline{v} & A \end{pmatrix} = \begin{pmatrix} 1 & \overline{0}^{\mathrm{T}} \\ \overline{0} & I \end{pmatrix}, \tag{15}$$

where I is the 3×3 identity matrix. By a straightforward calculation from this follow that the space-time transformation between the two inertial frames K and K′ is

$$\begin{pmatrix} t' \\ \overline{r} \end{pmatrix} = S_{\overline{v}} \begin{pmatrix} t \\ \overline{r'} \end{pmatrix} = \begin{pmatrix} \alpha & e\overline{v}^{\mathrm{T}} \\ \overline{v} & -\alpha P_{\overline{v}} - (I - P_{\overline{v}}) \end{pmatrix} \begin{pmatrix} t \\ \overline{r'} \end{pmatrix} \tag{16}$$

with α defined by

$$\alpha = \sqrt{1 - e|\overline{v}|^2}. \tag{17}$$

If we choose $\overline{v} = (v, 0, 0)$ and write $\overline{r} = (x, y, z)$ and $\overline{r'} = (x', y', z')$, the the above matrix become

$$S_{\overline{v}} = \begin{pmatrix} \alpha & ev & 0 & 0 \\ v & -\alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{18}$$

Solving $\left(\frac{t'}{\overline{r'}}\right)$ as a function of $\left(\frac{t}{\overline{r}}\right)$ we get

$$t' = \alpha^{-1}(t - evx) \qquad x' = \alpha^{-1}(vt - x) \qquad y' = -y \qquad z' = -z \tag{19}$$

which is the known Lorenz transformation (with space reversal) if $e = 1/c^2$.

3. SYMMETRIC VELOCITY AND INTERVAL CONSERVATION

In this section we will show that from the principle of relativity alone it follows that an interval is conserved, a ball of possible velocities is conserved and this ball is a bounded symmetric domain with respect to the projective maps. The symmetry of this ball, resulting from the above space-time transformations, fixes the so-called symmetric velocity.

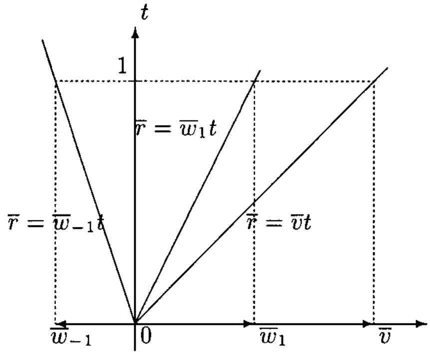

Fig. 3. Eigenspaces of the symmetry.

As mentioned above, the space-time transformation $S_{\bar{v}}$ between the frames K and K′ is a symmetry transformation. Such a symmetry is a reflection with respect to the set of the fixed points. We now want to determine the events fixed by $S_{\bar{v}}$, meaning that in both systems the event will have the same coordinates. From (16) it follows that such an event we have

$$\frac{\bar{r}'}{t} = \frac{\bar{v}}{1+\alpha} = \frac{\bar{r}}{t} := \bar{w}_1. \tag{20}$$

The meaning of this is that all the events fixed by the transformation $S_{\bar{v}}$ are on a straight world line through the origin of both frames at time $t = 0$ moving with velocity $\bar{w}_1$ (see Fig. 3) in both frames. Such velocity we will call a *symmetric velocity* between the frames K and K′. Similarly, for events in the plane generated by $\bar{v}$ and the t axis that are -1 eigenvectors of $S_{\bar{v}}$ we get:

$$\frac{\bar{r}'}{t} = \frac{\bar{v}}{\alpha - 1} = \frac{\bar{r}}{t} := \bar{w} - 1. \tag{21}$$

The symmetry $S_{\bar{v}}$ becomes an isometry if we introduce an appropriate inner product. Under this inner product the 1 and -1 eigenvectors of $S_{\bar{v}}$ must be orthogonal and the operator $S_{\bar{v}}$ become a self-adjoint operator. The new inner product may be obtained by leaving the inner product of the space components unchanged and introducing some weight μ for the time component. The orthogonality of $\bar{w}_1$ and $\bar{w}_{-1}$ imply $\mu^2 - \frac{1}{e} = 0$ and if $e > 0$, this implies

$$\mu = \frac{1}{\sqrt{e}}. \tag{22}$$

From the fact that S is an isometry with respect to the inner product with weight μ we have

$$(\mu t)^2 + |\bar{r}'|^2 = (\mu t')^2 + |\bar{r}|^2 \tag{23}$$

that is equivalent to

$$(\mu t')^2 + |\bar{r}'|^2 = (\mu t)^2 - |\bar{r}|^2. \tag{24}$$

This imply that our space-time transformation from K to K′ conserves the interval

$$ds^2 = (\mu dt)^2 - |d\bar{r}|^2 \tag{25}$$

with μ defined by (22).

Moreover, since the zero interval world lines are transformed by these transformations to zero interval lines this imply that the speed μ is conserved for any relativistic space-time transformation between two inertial systems. Obviously, also the cone $ds^2 > 0$, corresponding to a velocity ball D of radius μ, is preserved under this transformation. Thus, if we assume now conservation of the speed of light c, we get $e = 1/\mu^2 = 1/c^2$ and there is only one way of synchronizing the clocks by satisfying the principle of relativity.

From our space-time transformations (16) we can describe how to translate the velocity $\bar{u}'$ from frame K′ to corresponding velocity $\bar{u}$ in frame K. Direct calculation shows that the velocity transformation $s_{\bar{v}}$ between systems K and K′ is given by

$$\bar{u} = s_{\bar{v}}(\bar{u}') = \bar{v} - (\alpha^2 P_{\bar{v}} + \alpha(I - P_{\bar{v}}))\frac{\bar{u}'}{1 - e\langle \bar{v} \mid \bar{u}'\rangle} \tag{26}$$

If we perform a space reversion of K′, this will lead to $\bar{u}' \to -\bar{u}'$ leading to the known Einstein's velocity addition formula

$$\bar{u} = \varphi_{\bar{v}}(\bar{u}') = \bar{v} \oplus_E \bar{u}' = \bar{v} + (\alpha^2 P_{\bar{v}} + \alpha(I - P_{\bar{v}}))\frac{\bar{u}'}{1 + e\langle \bar{v} \mid \bar{u}'\rangle}. \tag{27}$$

From the definition of the symmetric velocity (20) we get

$$\bar{w}_1 \oplus_E \bar{w}_1 = \bar{v} \tag{28}$$

implying that the symmetric velocity is the relativistic half of the velocity $\bar{v}$. This shows that the ball D of possible velocities is a symmetric domain, since for any velocity $\bar{w}_1$ we can define $\bar{v}$ by (28). Then $s_{\bar{v}}$ defined by (26) is a symmetry fixing only $\bar{w}_1$. Note that since $s_{\bar{v}}$ was obtained by restricting of a linear map $S_{\bar{v}}$ to a hyperplane $t = 1$ it follows that $s_{\bar{v}}$ is a projective map (meaning line segment preserving).

4. SYMMETRIC VELOCITY, CONFORMAL GROUP, AND SPIN TRIPLE PRODUCT

In previous section we saw that the symmetric velocity plays an important role in the relativistic transformations between two inertial systems. The symmetric velocity was introduced in Friedman and Naimark (1992) in order to transform the the transformations $\varphi_{\bar{v}}$ from (27) to conformal ones. We will assume from now conservation of speed of light. Then the symmetric velocity $\bar{w}$ and velocity $\bar{v}$ are connected by

$$\bar{w} = F(\bar{v}) = \frac{\bar{v}}{1 + \sqrt{1 - |\bar{v}|^2/c^2}}, \qquad \bar{v} = F^{-1}(\bar{w}) = \frac{2\bar{w}}{1 + |\bar{w}|^2/c^2}. \tag{29}$$

As mentioned above, the symmetric velocity is the relativistic half of the regular velocity. The set of all possible symmetric velocities in any inertial frame form a 3D ball S of radius c. For simplicity of notation we will assume from now $c = 1$. As it was shown the map $\psi = F\varphi F^{-1}$ is a conformal map of the ball. A similar result was obtained by A. Ungar in 1996 (see Ungar, 2001) in the study of so-called Möbius gyrovector space.

An explicit form of the conformal map of the ball was proposed by Ahlfors in 1981 and is given by the formula (see Ungar, 2001) for the extended Möbius transformation

$$\psi_{\bar{u}}(\bar{w}) = \bar{u} \oplus \bar{w} = \frac{(1 + 2\langle \bar{u} \mid \bar{w}\rangle + |\bar{w}|^2)\bar{u} + (1 - |\bar{u}|^2)\bar{w}}{1 + 2\langle \bar{u} \mid \bar{w}\rangle + |\bar{u}|^2|\bar{w}|^2}, \quad \bar{u}, \bar{w} \in S \tag{30}$$

where $\oplus$ will denote the sum of symmetric velocities.

If the evolution of the system is described by conformal maps of the symmetric velocities, the dynamic equation will involve the generators of such maps. In order to obtain the generators of the boosts we have to take a one-parameter family $\psi_{\bar{u}(\tau)}$ depending on some real parameter τ with $\bar{u}(0) = 0$. Then the generator is given by

$$\xi_{\bar{a}}(\bar{w}) = \frac{d}{d\tau}\, \psi_{\bar{u}(\tau)}(\bar{w})|_{\tau=0} = a - 2\langle \bar{w} \mid \bar{a}\rangle\bar{w} + |\bar{w}|^2\bar{a} \tag{31}$$

with $\bar{a} = \frac{d}{d\tau}\bar{u}(\tau)|_{\tau=0}$. This is a general formula for the generators of translations in the Lie algebra of a bounded symmetric domain (see Loos, 1977). As mentioned above, the generators of the translations are of the form

$$\xi_{\bar{a}}(\bar{w}) = \bar{a} - \{\bar{w}, \bar{a}, \bar{w}\}, \tag{32}$$

with $\{\bar{w}, \bar{a}, \bar{w}\}$ as the triple product associated with the bounded symmetric domain. In our case the domain is a real domain of type 4 and dimension 3 in Cartan's classification, called also the Spin factor with the triple product defined by (5). The norm is the usual Euclidean norm. The ball of radius one is homogeneous under the group of conformal maps (30).

The Lie algebra of the conformal group consists of generators of boosts described by (31) and (32) in terms of the triple product and of generators of rotations. To describe the generators of rotations in the symmetric velocity ball we chose first an orthonormal basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ in R_3 and define

$$\bar{D} = (D(\bar{e}_2, \bar{e}_3),\ D(\bar{e}_3, \bar{e}_1),\ D(\bar{e}_1, \bar{e}_2)),$$

where the linear operator $D(a, b)$ is defined by (2). Then the generator of rotation for any symmetric velocity $\overline{w}$ is expressed in the triple product by

$$\vartheta(\overline{w}) = (H \cdot \overline{D})(\overline{w}) = H \times \overline{w} : H \in R^3.$$

Thus, any element ζ of our Lie algebra is of the form

$$\zeta = \zeta_{\overline{a},H}(\overline{w}) = \overline{a} + (H \cdot \overline{D})(\overline{w}) - \{\overline{w}, \overline{a} \cdot \overline{w}\}\, \overline{a},\ H \in R^3 \tag{33}$$

and is expressed in terms of the triple product.

Note that also (30) could be obtained in terms of the triple product by exponentiating (32) and using the explicit form for this exponent from Kaup (1983).

5. RELATIVISTIC DYNAMIC EQUATIONS USING SYMMETRIC VELOCITY

We are going to describe now how to obtain the relativistic dynamic equation for the symmetric velocities. From the definition of the symmetric velocity

$$\gamma = \frac{1}{\sqrt{1 - |\overline{v}|^2}} = \frac{1 + |\overline{w}|^2}{1 - |\overline{w}|^2}. \tag{34}$$

and thus

$$mv = m_0 \gamma v = m_0 \frac{2\overline{w}}{1 - |\overline{w}|^2} \tag{35}$$

with m_0 the rest mass of the object.

The relativistic dynamic equation

$$F = \frac{d}{dt}(m\overline{v})$$

for the symmetric velocities now becomes

$$F = \frac{d}{dt} m_0 \frac{2\overline{w}}{1 - |\overline{w}|^2} = 2m_0 \left(\frac{1}{1 - |\overline{w}|^2} \frac{d\overline{w}}{dt} + \frac{2\overline{w}}{(1 - |\overline{w}|^2)^2} \left\langle \frac{d\overline{w}}{dt} | \overline{w} \right\rangle \right). \tag{36}$$

By taking the inner product with $\overline{w}$ we get

$$\langle F \mid \overline{w} \rangle = 2m_0 \left\langle \frac{d\overline{w}}{dt} | \overline{w} \right\rangle \frac{1 + |\overline{w}|^2}{(1 - |\overline{w}|^2)^2}. \tag{37}$$

By substituting $\langle \frac{d\overline{w}}{dt} \mid \overline{w} \rangle$ from (37) into (36) we obtain

$$\frac{2m_0}{1 - |\overline{w}|^2} \frac{d\overline{w}}{dt} = F - \frac{2\overline{w}}{1 + |\overline{w}|^2} \langle F \mid \overline{w} \rangle. \tag{38}$$

Multiplying both sides of (38) by $1 + |\overline{w}|^2$ and using that $dt = \gamma dr$ we obtain *the relativistic dynamic equation for the symmetric velocities*

$$2m_0 \frac{d\overline{w}}{d\tau} = F - c^{-2}\{\overline{w}, F, \overline{w}\} = \xi_F(\overline{w}), \tag{39}$$

where τ denotes the proper time and the triple product is the spin triple product defined by (5) and ξ by (31) and (32). Thus, a force F that does not have linear dependence on $\overline{v}$ will generate a conformal flow on the ball S representing the symmetric velocities.

Let us derive now the relativistic dynamic equation for the electromagnetic field for the symmetric velocities. Let E denotes the electric strength of the field, H denote the magnetic strength. Then from the formula of the Lorentz force for the electromagnetic field, the dynamic equation becomes

$$\frac{d}{dt}(m\overline{v}) = q(E + \overline{v} \times H).$$

Thus, using Eqs. (34) and (35) we get

$$\begin{aligned} q&\left(E + \frac{2\overline{w}}{1+|\overline{w}|^2} \times H\right) \\ &= \frac{d}{dt} m_0 \frac{2\overline{w}}{1-|\overline{w}|^2} \\ &= 2m_0 \left(\frac{1}{1-|\overline{w}|^2} \frac{d\overline{w}}{dt} + \frac{2\overline{w}}{(1-|\overline{w}|^2)^2} \left\langle \frac{d\overline{w}}{dt} |\overline{w} \right\rangle \right). \end{aligned} \tag{40}$$

By taking the inner product with w we get

$$q\langle E \mid \overline{w}\rangle = 2m_0 \left\langle \frac{d\overline{w}}{dt} |\overline{w} \right\rangle \frac{1+|\overline{w}|^2}{1-|\overline{w}|^2}. \tag{41}$$

By substituting $\langle \frac{d\overline{w}}{dt} \mid \overline{w}\rangle$ from (41) into (40) we obtain

$$\frac{2m_0}{1-|\overline{w}|^2} \frac{d\overline{w}}{dt} = q \left(E + \frac{2\overline{w}}{1+|\overline{w}|^2} \times H - \frac{2\overline{w}}{1+|\overline{w}|^2} \langle E \mid \overline{w}\rangle \right). \tag{42}$$

Multiplying both sides of the last equation by $1 + |\overline{w}|^2$ and using that $dt = \gamma d\tau$ we obtain

$$2m_0 d\overline{w}/d\tau = q(E - c^{-2}\{\overline{w}, E, \overline{w}\} + 2\overline{w} \times H) \tag{43}$$

that could be considered as the relativistic dynamic equation for the electromagnetic field. Thus, also the electromagnetic field generates a conformal flow on the ball S representing the symmetric velocities.

6. THE PHYSICAL MEANING OF THE GEOMETRIC PRODUCT

In the last decades it was found that the use of Clifford algebra and the geometric product (see Hestenes and Sobczyk, 1984) associated with it simplify the description of different physical phenomena in Classical and Modern physics. In order to represent this product the physical quantities were imbedded into the Clifford algebra. But what is the reason that this description is so successful? So far not much is known why this algebraic structure is connected with the description of real phenomena. Often, it is proposed that the success in use of this algebraic structure is connected with the fact that the geometric product contains the dot product and the outer (vector) product.

As it was shown in Section 2, the Lorenz transformation describing the space-time transformation between two inertial systems could be obtained from the principle of relativity alone. In Section 3 it is shown that such transformation is a symmetry with respect a world line defined by the symmetric velocity, expressing the relative motion of the systems. As shown in Section 4, the Lorenz group acts on the ball S of all possible symmetric velocities by conformal maps. The Lie algebra of the conformal group is fully described by the spin triple product. In Section 5 it was shown that the relativistic dynamic equations in mechanics and for the electromagnetic force involve only spin triple product and define a conformal flow on S.

For a real spin factor or equivalently for a Cartan domain of type 4 from (5) follow that the spin triple product is

$$\{a, b, c\} = D(a, b)c = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, c\rangle b. \tag{44}$$

Thus, the operator $D(u, v)$ is equal to

$$D(u, v) = \langle u \mid v\rangle I + u \wedge v, \tag{45}$$

where I denotes the identity operator. This is similar to the geometric product between u and v defined as

$$uv = \langle u \mid v\rangle + u \wedge v, \tag{46}$$

where the Sum of a scalar $\langle u \mid v\rangle$ and bivector $u \wedge v$ makes sense in the Clifford algebra. Therefore, the operator *D(u,v)* represents the geometric product as natural operator on the spin factor.

We want to mention the following difference between the two representations of the geometric product: the first one as the product in the Clifford algebra and the second one as operators of the spin triple product. In the first case, in order to represent a vector space of dimension n we need an algebra of dimension 2^n, while in the second case it is enough to consider the same n-dimensional vector space with the spin triple product on it. The spin factor and the spin triple product result from basic principles in physics. The spin triple product is build directly from the conformal group in real case and in complex case it is built totally on geometry of Cartan domain of type 4 representing two-state systems in Quantum Mechanics (see Friedman and Russo, 1992). The Lorentz group is represented in both cases by a spin half representation. The spin factor has also the spin one representation and the representation of the canonical anticommutation relations as shown in Friedman and Russo (2001). As any bounded symmetric domain, the spin factor has an explicitly defined invariant measure, Harmonic analysis, spectral theorem, quantization, and representation as operators on a Hilbert space. But, since the second representation is more compact, currently we are missing several techniques that played important role in the Clifford algebra approach. For instance, here we do not have multivectors of order 3 and higher and we do not have the analog of the I operator. But we believe that these difficulties could be overcomed.

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